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The Foldy–Wouthuysen representation revisited

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Abstract. On the basis of the approach of Eriksen and Kolsrud, we derive a canonical transformation which diagonalises the Dirac Hamiltonian for a particle interacting with an external electromagnetic field. We show that an explicitly gauge invariant electromagnetic interaction can be obtained without recourse to the $1/m$ expansion. When expanded in $1/m$, our result coincides with the Foldy-Wouthuysen interaction.

1. Introduction

In 1950 Foldy and Wouthuysen [1] first showed that it is possible to diagonalise the Dirac Hamiltonian for a fermion interacting with an external electromagnetic field. By means of successive canonical transformations order by order in the reciprocal of the mass m, they found a representation in which the positive- and negative-energy components are decoupled. The Foldy-Wouthuysen representation is advantageous in interpreting the physical content of the Dirac equation in conformity with classical electrodynamics. Among other things it preserves the invariance under the local gauge transformation. Its virtue, however, is somewhat diminished by the fact that the transformation is not known in a closed form. When one goes to higher orders in $1/m$, the procedure becomes intractably complicated.

On the other hand, Eriksen and Kolsrud [2] have proposed a different expansion scheme in powers of the electric charge e. The diagonalisation of the Dirac Hamiltonian is accomplished order by order in e. The advantage of their approach is that the $1/m$ expansion is not required. Their final expression, when expanded in $1/m$, however, does not coincide with that of Foldy and Wouthuysen. Thus the gauge invariance does not manifest itself in Eriksen and Kolsrud.

The disagreement between the two approaches is not surprising. There was much controversy about the ambiguity in diagonalising the external-field Dirac Hamiltonians [3-6]. It is now well recognised that the canonical transformation which brings the Dirac Hamiltonian into an even form cannot be determined uniquely. The fact is that long before the debate began, Eriksen and Kolsrud had noticed the non-uniqueness of the transformation and presented various unitary operators which, as they admit, are different from Foldy and Wouthuysen. Barnhill [3] argued that the requirement of gauge invariance leads to a unique choice for the canonical transformation. However, in principle it is possible to use a representation which is not gauge invariant explicitly.

Moreover, a question was raised as to the gauge invariance of the Foldy-Wouthuysen interaction. Nieto [4] and Goldman [5] pointed out that the explicit gauge invariance of the transformed Hamiltonian should not be expected in the time-dependent external electromagnetic field problem. In fact, when one separates the entire coupled-field Hamiltonian into a particle Hamiltonian and a radiation field

Hamiltonian, the transformed particle Hamiltonian cannot be equivalent to the original Dirac Hamiltonian in the sense that the energy expectation values are not the same in the two representations. This does not invalidate the attractive feature of the Foldy-Wouthuysen representation. As suggested by Woloshyn [6], the Foldy-Wouthuysen interaction gives the correct results when one uses it in the calculation of the S matrix. Thus the explicit gauge invariance of the Foldy-Wouthuysen interaction, if interpreted correctly, remains a strong point.

To summarise the above discussion, we find that a canonical transformation is proposed by Eriksen and Kolsrud in a closed form but in a gauge-variant way, whereas the gauge-invariant Foldy-Wouthuysen interaction is known only in an infinite series of $1/m$ expansion. The purpose of this paper is to derive a canonical transformation which furnishes us with an explicitly gauge-invariant electromagnetic interaction. We do not use a $1/m$ expansion, however. In § 2 we start from the brief review of the approach of Eriksen and Kolsrud and in § 3 we seek the canonical transformation that makes the Eriksen-Kolsrud interaction manifestly gauge invariant. In § 4 our new interaction is compared with the Foldy-Wouthuysen representation. Finally in § 5 we discuss the implications of our approach.

2. The approach of Eriksen and Kolsrud

Consider the Dirac equation

$$\mathrm{i}(\partial/\partial t)\psi_{\mathrm{D}} = H_{\mathrm{D}}\psi_{\mathrm{D}} = (H_0 + eH_1)\psi_{\mathrm{D}} \tag{2.1}$$

where H_0 is the free particle Hamiltonian

$$H_0 = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m \tag{2.2}$$

and H_1 is the electromagnetic interaction

$$H_1 = -\boldsymbol{\alpha} \cdot \boldsymbol{A} + \phi \tag{2.3}$$

due to the external vector and scalar potentials $\boldsymbol{A}$ and ϕ evaluated at the position of the particle. An arbitrary transformation U converts H_{D} into

$$H = UH_{\mathrm{D}}U^{-1} - \mathrm{i}U(\partial/\partial t)U^{-1}. \tag{2.4}$$

The requirement that the new Hamiltonian H be an even operator is

$$[\beta, H] = 0 \tag{2.5}$$

i.e.

$$\mathrm{i}(\partial/\partial t)\Lambda + [\Lambda, H_{\mathrm{D}}] = 0. \tag{2.6}$$

Here Λ is defined by

$$\Lambda = U^{-1}\beta U. \tag{2.7}$$

Eriksen and Kolsrud expanded Λ in powers of e:

$$\Lambda = \Lambda_0 + e\Lambda_1 + e^2\Lambda_2 + \dots . \tag{2.8}$$

The first term Λ_0 corresponds to the unitary transformation

$$U_0 = [2E(E+m)]^{-1/2}(E + \beta H_0) \tag{2.9}$$

which diagonalises the free particle Hamiltonian H_0. E is the operator

$$E=(p^2+m^2)^{1/2}. \tag{2.10}$$

Using (2.9) Λ_0 is given by

$$\Lambda_0=E^{-1}H_0. \tag{2.11}$$

$\Lambda_1, \Lambda_2, \ldots$ are determined by the equations

$$\mathrm{i}(\partial/\partial t)\Lambda_1+[\Lambda_1, H_0]+[\Lambda_0, H_1]=0 \tag{2.12}$$

etc. Equation (2.12) can be solved formally:

$$\Lambda_1^c=0 \tag{2.13}$$

$$\Lambda_1^a=2\mathrm{i}\Lambda_0\int_{-\infty}^{t}\mathrm{d}t'\exp[\mathrm{i}H_0(t'-t)]H_1^a\exp[-\mathrm{i}H_0(t'-t)]. \tag{2.14}$$

H_1^a in the integrand should be evaluated at time t'. Use was made of the fact that any operator $\mathcal{O}$ can be decomposed into $\mathcal{O}^c$ and $\mathcal{O}^a$:

$$\mathcal{O}^c=\tfrac{1}{2}(\mathcal{O}+\Lambda_0\mathcal{O}\Lambda_0) \tag{2.15}$$

$$\mathcal{O}^a=\tfrac{1}{2}(\mathcal{O}-\Lambda_0\mathcal{O}\Lambda_0). \tag{2.16}$$

Eriksen and Kolsrud proposed various solutions to the canonical transformation U. Here we only refer to the transformation in the closed form

$$U=U_0(\Lambda_0\Lambda)^{1/2}. \tag{2.17}$$

This brings us to the new Hamiltonian

$$H=\beta E+U_0\{eH_1^c+\tfrac{1}{4}e^2\Lambda_0[\Lambda_1^a, H_1^a]_+ +\tfrac{1}{4}e^3\Lambda_0[\Lambda_2^a, H_1^a]_+ +\ldots\}U_0^{-1}. \tag{2.18}$$

It is instructive to see that a further transformation

$$\exp(-\tfrac{1}{8}e^3[\Lambda_1^a, \Lambda_2^a]) \tag{2.19}$$

yields

$$\beta E+U_0\{eH_1^c+\tfrac{1}{4}e^2\Lambda_0[\Lambda_1^a, H_1^a]_+ -\tfrac{1}{8}e^3[\Lambda_1^a,[\Lambda_1^a, H_1^c]]+\ldots\}U_0^{-1}. \tag{2.20}$$

This Hamiltonian was derived by Eriksen and Kolsrud using a different iterative method from ours. The existence of the equivalent Hamiltonians exemplifies the fact that diagonalisation of the Dirac equation is not unique.

3. Restoration of gauge invariance

We define

$$S^{(-)}=-\tfrac{1}{2}\mathrm{i}U_0\Lambda_0\Lambda_1^aU_0^{-1}. \tag{3.1}$$

From (2.14) it follows that

$$S^{(-)}=\int_{-\infty}^{t}\mathrm{d}t'\exp[\mathrm{i}\beta E(t'-t)]H_1^{(-)}\exp[-\mathrm{i}\beta E(t'-t)]. \tag{3.2}$$

Either of the Hamiltonians (2.18) or (2.20) is written as

$$H=\beta E+eH_1^{(+)}+\tfrac{1}{2}\mathrm{i}e^2[S^{(-)}, H_1^{(-)}]+\ldots \tag{3.3}$$

where

$$H_1^{(+)} = U_0 H_1^c U_0^{-1} \tag{3.4}$$

$$H_1^{(-)} = U_0 H_1^a U_0^{-1}. \tag{3.5}$$

We write them down explicitly:

$$H_1^{(+)} = [2E(E+m)]^{-1/2}\{(E+m)\phi(E+m) + \boldsymbol{\alpha}\cdot\boldsymbol{p}\phi\boldsymbol{\alpha}\cdot\boldsymbol{p} - \beta\boldsymbol{\alpha}\cdot\boldsymbol{p}\boldsymbol{\alpha}\cdot\boldsymbol{A}(E+m) + (E+m)\boldsymbol{\alpha}\cdot\boldsymbol{A}\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}\}[2E(E+m)]^{-1/2} \tag{3.6}$$

$$H_1^{(-)} = [2E(E+m)]^{-1/2}[\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}\phi(E+m) - (E+m)\phi\beta\boldsymbol{\alpha}\cdot\boldsymbol{p} - (E+m)\boldsymbol{\alpha}\cdot\boldsymbol{A}(E+m) + \boldsymbol{\alpha}\cdot\boldsymbol{p}\boldsymbol{\alpha}\cdot\boldsymbol{A}\boldsymbol{\alpha}\cdot\boldsymbol{p}][2E(E+m)]^{-1/2}. \tag{3.7}$$

The gauge-invariant Hamiltonian must be in the form

$$e\phi + \beta[(\boldsymbol{p}-e\boldsymbol{A})^2 + m^2]^{1/2} + (\text{functions of } \boldsymbol{p}-e\boldsymbol{A}, \boldsymbol{B} \text{ and } \boldsymbol{E}) \tag{3.8}$$

where $\boldsymbol{B}$ and $\boldsymbol{E}$ are the magnetic and electric fields, respectively,

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \tag{3.9}$$

$$\boldsymbol{E} = -\nabla\phi - (\partial/\partial t)\boldsymbol{A}. \tag{3.10}$$

When expanded in powers of e, the third term of (3.8) begins with linear functions of $\boldsymbol{B}$ and $\boldsymbol{E}$, while the second term is written as

$$\beta[(\boldsymbol{p}-e\boldsymbol{A})^2 + m^2]^{1/2} = \beta E + eD_1 + e^2 D_2 + \dots. \tag{3.11}$$

Squaring both sides we find

$$[\beta E, D_1]_+ = -\boldsymbol{p}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{p} \tag{3.12}$$

$$[\beta E, D_2]_+ = \boldsymbol{A}^2 - D_1^2. \tag{3.13}$$

Using expression (3.11) we rewrite (3.3) as

$$H = e\phi + \beta[(\boldsymbol{p}-e\boldsymbol{A})^2 + m^2]^{1/2} + e(H_1^{(+)} - \phi - D_1) + e^2\{\tfrac{1}{2}\mathrm{i}[S^{(-)}, H_1^{(-)}] - D_2\} + \dots. \tag{3.14}$$

The main task of this paper is to see how one can make (3.14) compatible with (3.8). To this end we define Γ through

$$H_1^{(+)} - \phi - D_1 = [2E(E+m)]^{-1/2}\tfrac{1}{2}\Gamma[2E(E+m)]^{-1/2}. \tag{3.15}$$

After some manipulations we are led to

$$\Gamma = -\beta[E+m, \boldsymbol{\sigma}\cdot\boldsymbol{B}]_+ - \nabla\cdot\boldsymbol{F} + \boldsymbol{\sigma}\cdot(\boldsymbol{p}\times\boldsymbol{F} - \boldsymbol{F}\times\boldsymbol{p}) - \beta[E+m, \boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}]_+ - 2[2E(E+m)]^{1/2}D_1[2E(E+m)]^{1/2} + [E+m, E\phi + \phi E]_+ - 2[2E(E+m)]^{1/2}\phi[2E(E+m)]^{1/2}. \tag{3.16}$$

$\boldsymbol{F}$ stands for the quantity

$$\boldsymbol{F} = -\nabla\phi + \mathrm{i}[\beta E, \boldsymbol{A}]. \tag{3.17}$$

As Eriksen and Kolsrud noticed, $\boldsymbol{F}$ can be converted to $\boldsymbol{E}$ by means of a unitary transformation. Therefore the first line in (3.16) could be made consistent with the requirement of gauge invariance, as we will see later.

The most crucial step we take is to introduce an operator L in such a way that

$$\mathrm{i}[E^2, L\phi] = [E+m, E\phi + \phi E]_+ - 2[2E(E+m)]^{1/2}\phi[2E(E+m)]^{1/2}. \tag{3.18}$$

L is a very complicated operator which contains $\boldsymbol{p}$ both on the left-hand and right-hand sides of ϕ. It is most easily expressed in momentum space as given in the appendix, but its explicit form is not necessary for our discussion that follows. Recalling the definition of E we can write (3.18) as

$$\mathrm{i}[E^2, L\phi] = \mathrm{i}[\boldsymbol{p}^2, L\boldsymbol{\phi}] = L(\boldsymbol{p}\cdot\nabla\phi + \nabla\phi\cdot\boldsymbol{p}). \tag{3.19}$$

Next we make the replacement $\phi \to -\boldsymbol{p}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{p}$ in (3.18). The right-hand side of (3.18) becomes

$$\begin{aligned}-[E+m, [E, \boldsymbol{p}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{p}]_+]_+ + 2[2E(E+m)]^{1/2}(\boldsymbol{p}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{p})[2E(E+m)]^{1/2}\\ = [\beta E, -\beta[E+m, \boldsymbol{p}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{p}]_+ - 2[2E(E+m)]^{1/2}D_1[2E(E+m)]^{1/2}]_+\end{aligned} \tag{3.20}$$

where we have used (3.12). On the other hand, the left-hand side of (3.18) turns out to be

$$-\mathrm{i}[E^2, L(\boldsymbol{p}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{p})] = -\mathrm{i}[\beta E, [\beta E, L(\boldsymbol{p}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{p})]]_+. \tag{3.21}$$

Comparing (3.20) with (3.21) we obtain

$$\begin{aligned}-\beta[E+m, \boldsymbol{p}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{p}]_+ - 2[2E(E+m)]^{1/2}D_1[2E(E+m)]^{1/2}\\ = -\mathrm{i}[\beta E, L(\boldsymbol{p}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{p})].\end{aligned} \tag{3.22}$$

It is pleasing to find that upon inserting (3.18) and (3.22) into (3.16) Γ contains only $\boldsymbol{B}$ and $\boldsymbol{F}$:

$$\Gamma = -\beta[E+m, \boldsymbol{\sigma}\cdot\boldsymbol{B}]_+ - \nabla\cdot\boldsymbol{F} - \boldsymbol{p}\cdot(\boldsymbol{\sigma}\times\boldsymbol{F}+L\boldsymbol{F}) - (\boldsymbol{\sigma}\times\boldsymbol{F}+L\boldsymbol{F})\cdot\boldsymbol{p}. \tag{3.23}$$

As a consequence we can extract a gauge-invariant part $H_1^{(+)'}$:

$$H_1^{(+)} = H_1^{(+)'} - \mathrm{i}[S, \beta E] + (\partial/\partial t)S \tag{3.24}$$

where

$$H_1^{(+)'} = \phi + D_1 + [2E(E+m)]^{-1/2}\tfrac{1}{2}\Gamma'[2E(E+m)]^{-1/2} \tag{3.25}$$

$$\Gamma' = -\beta[E+m, \boldsymbol{\sigma}\cdot\boldsymbol{B}]_+ - \nabla\cdot\boldsymbol{E} - \boldsymbol{p}\cdot(\boldsymbol{\sigma}\times\boldsymbol{E}+L\boldsymbol{E}) - (\boldsymbol{\sigma}\times\boldsymbol{E}+L\boldsymbol{E})\cdot\boldsymbol{p} \tag{3.26}$$

and

$$S = -\tfrac{1}{2}[2E(E+m)]^{-1/2}[\nabla\cdot\boldsymbol{A} + \boldsymbol{p}\cdot(\boldsymbol{\sigma}\times\boldsymbol{A}+L\boldsymbol{A}) + (\boldsymbol{\sigma}\times\boldsymbol{A}+L\boldsymbol{A})\cdot\boldsymbol{p}][2E(E+m)]^{-1/2}. \tag{3.27}$$

We return to (3.3) and exploit the freedom to transform H by an arbitrary unitary transformation $\exp(\mathrm{i}eS^{(+)})$

$$H' = \mathrm{e}^{\mathrm{i}eS^{(+)}}H\,\mathrm{e}^{-\mathrm{i}eS^{(+)}} - \mathrm{i}\,\mathrm{e}^{\mathrm{i}eS^{(+)}}(\partial/\partial t)\,\mathrm{e}^{-\mathrm{i}eS^{(+)}}. \tag{3.28}$$

Expanded in powers of e, (3.28) becomes

$$\begin{aligned}H' = \beta E + eH_1^{(+)} + \mathrm{i}e[S^{(+)}, \beta E] - e(\partial/\partial t)S^{(+)} + \tfrac{1}{2}\mathrm{i}e^2[S^{(-)}, H_1^{(-)}] + \mathrm{i}e^2[S^{(+)}, H_1^{(+)}]\\ + \tfrac{1}{2}\mathrm{i}e^2[S^{(+)}, \mathrm{i}[S^{(+)}, \beta E] - (\partial/\partial t)S^{(+)}] + \dots\end{aligned} \tag{3.29}$$

Undesired terms in (3.24) can be eliminated by choosing $S^{(+)} = S$. That is

$$\begin{aligned}H' = \beta E + eH_1^{(+)'} + \tfrac{1}{2}\mathrm{i}e^2[S^{(-)}, H_1^{(-)}] + \tfrac{1}{2}\mathrm{i}e^2[S^{(+)}, H_1^{(+)}]\\ + \tfrac{1}{2}\mathrm{i}e^2[S^{(+)}, H_1^{(+)'}] + \dots\end{aligned} \tag{3.30}$$

Alternatively this can be cast in the form

$$H' = e\phi + \beta[(\boldsymbol{p} - e\boldsymbol{A})^2 + m^2]^{1/2} + eH_1' + e^2 H_2' + \ldots \tag{3.31}$$

where

$$H_1' = [2E(E+m)]^{-1/2} \tfrac{1}{2}\Gamma' [2E(E+m)]^{-1/2} \tag{3.32}$$

$$H_2' = \tfrac{1}{2}\mathrm{i}[S^{(-)}, H_1^{(-)}] + \tfrac{1}{2}\mathrm{i}[S^{(+)}, H_1^{(+)}] + \tfrac{1}{2}\mathrm{i}[S^{(+)}, H_1' + \phi + D_1] - D_2. \tag{3.33}$$

4. Comparison with Foldy and Wouthuysen

Our electromagnetic interaction (3.31) is given in a power series of e but the $1/m$ expansion is not used. It is interesting to expand (3.31) in $1/m$ and compare with the Foldy-Wouthuysen interaction. For this purpose we first derive the operator L given by (3.18). Expanding the right-hand side of (3.18) in $1/m$ we obtain

$$L\phi = -\frac{5}{16m^2}(\boldsymbol{p} \cdot \nabla\phi + \nabla \cdot \phi\boldsymbol{p}). \tag{4.1}$$

The interaction H_1' to the required order $1/m^2$ is not affected by L so that

$$H_1' = -\frac{1}{2m}\beta\boldsymbol{\sigma} \cdot \boldsymbol{B} - \frac{1}{8m^2}\nabla \cdot \boldsymbol{E} - \frac{1}{8m^2}(\boldsymbol{p} \cdot \boldsymbol{\sigma} \times \boldsymbol{E} + \boldsymbol{\sigma} \times \boldsymbol{E} \cdot \boldsymbol{p}). \tag{4.2}$$

Next consider e^2 terms. The first term in (3.33) comes from creation of a virtual particle-antiparticle pair and subsequent annihilation. Its derivation is tedious but straightforward. To order $1/m^2$

$$\tfrac{1}{2}\mathrm{i}[S^{(-)}, H_1^{(-)}] = \frac{\beta}{2m}\boldsymbol{A}^2 + \frac{1}{4m^2}\boldsymbol{\sigma} \cdot \boldsymbol{A} \times \frac{\partial}{\partial t}\boldsymbol{A} + \frac{1}{2m^2}\boldsymbol{\sigma} \cdot \boldsymbol{A} \times \nabla\phi. \tag{4.3}$$

From (3.28) $S^{(+)} = S$ is given by

$$S^{(+)} = -\frac{1}{8m^2}(\nabla \cdot \boldsymbol{A} + \boldsymbol{p} \cdot \boldsymbol{\sigma} \times \boldsymbol{A} + \boldsymbol{\sigma} \times \boldsymbol{A} \cdot \boldsymbol{p}) \tag{4.4}$$

so that

$$\tfrac{1}{2}\mathrm{i}[S^{(+)}, H_1^{(+)}] = -\frac{1}{8m^2}\boldsymbol{\sigma} \cdot \boldsymbol{A} \times \nabla\phi \tag{4.5}$$

$$\tfrac{1}{2}\mathrm{i}[S^{(+)}, H_1' + \phi + D_1] = -\frac{1}{8m^2}\boldsymbol{\sigma} \cdot \boldsymbol{A} \times \nabla\phi. \tag{4.6}$$

Note that H_1' and

$$D_1 = -\frac{\beta}{2m}(\boldsymbol{A} \cdot \boldsymbol{p} + \boldsymbol{p} \cdot \boldsymbol{A}) \tag{4.7}$$

did not contribute to (4.6). Finally, D_2 is given by

$$D_2 = \frac{\beta}{2m}\boldsymbol{A}^2. \tag{4.8}$$

Collecting (4.3), (4.5), (4.6) and (4.8) we get

$$H_2' = -\frac{1}{4m^2}\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{E}. \tag{4.9}$$

Our final expression is

$$H' = \beta m + \frac{\beta}{2m}(\boldsymbol{p}-e\boldsymbol{A})^2 + e\phi - \frac{e}{2m}\beta\boldsymbol{\sigma}\cdot\boldsymbol{B} + \frac{e}{8m^2}[\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A})\times\boldsymbol{E} - \boldsymbol{\sigma}\cdot\boldsymbol{E}\times(\boldsymbol{p}-e\boldsymbol{A})] - \frac{e}{8m^2}\nabla\cdot\boldsymbol{E}. \tag{4.10}$$

This exactly reproduces the Foldy-Wouthuysen interaction. It should be emphasised that the unitary transformation which made H_1' gauge invariant naturally entails a modification of H_2', thereby leading to the Foldy-Wouthuysen transformation.

5. Discussion

Starting from the Eriksen-Kolsrud interaction we derived a new representation of the electromagnetic interaction of a Dirac particle. If we are concerned with a structureless point particle, extraction of the explicitly gauge-invariant single-particle Hamiltonian from the total coupled-field Hamiltonian is only an academic problem. In fact, the covariant perturbation scheme of quantum electrodynamics has been enormously successful. Nothing new is gained from our non-covariant Hamiltonian.

The situation is different for a composite system consisting of constituent fermions. In the limit when the internal interaction among the constituent particles vanishes the Hamiltonian of the composite system in the presence of an external field should be a sum of the diagonalised single-particle Hamiltonians. When the internal interaction is switched on, we have to add the terms which depend directly on the internal interaction. If we do not use the explicitly gauge-invariant single-particle Hamiltonian, current conservation is fulfilled only after we calculate the internal interaction terms consistently with the single-constituent term. This is an awkward procedure. It is more the case when there is no underlying well-established theory for the interaction among the constituents. It is desired that both terms are gauge invariant separately. Our electromagnetic interaction as well as the Foldy-Wouthuysen interaction will be useful in this respect.

Appendix

We sandwich the left-hand and right-hand sides of (3.18) between two plane wave states $|\boldsymbol{k}\rangle$ and $|\boldsymbol{k}'\rangle$:

$$\mathrm{i}(E_{k'}^2 - E_k^2)\langle\boldsymbol{k}'|L\phi|\boldsymbol{k}\rangle = \{(E_{k'}+E_k)(E_{k'}+E_k+2m) - 2[2E_{k'}(E_{k'}+m)]^{1/2}[2E_k(E_k+m)]^{1/2}\}\langle\boldsymbol{k}'|\phi|\boldsymbol{k}\rangle \tag{A1}$$

where

$$E_k = (\boldsymbol{k}^2+m^2)^{1/2} \qquad E_{k'} = (\boldsymbol{k}'^2+m^2)^{1/2}. \tag{A2}$$

From (A1) we can write as

$$\langle \boldsymbol{k}'|L\phi|\boldsymbol{k}\rangle = L_{\boldsymbol{k}'\boldsymbol{k}}\langle \boldsymbol{k}'|\phi|\boldsymbol{k}\rangle \tag{A3}$$

$$L_{\boldsymbol{k}'\boldsymbol{k}} = -\mathrm{i}\,\frac{E_{k'} - E_k}{E_{k'} + E_k} \times \frac{(E_{k'} + E_k + 2m)^2 + 4E_{k'}E_k}{(E_{k'} + E_k)(E_{k'} + E_k + 2m) + 2[2E_{k'}(E_{k'} + m]^{1/2}[2E_k(E_k + m)]^{1/2}}. \tag{A4}$$

To lowest order in $1/m$ expansion

$$L_{\boldsymbol{k}'\boldsymbol{k}} \approx -\frac{5}{16m^2}\,\mathrm{i}(\boldsymbol{k}'^2 - \boldsymbol{k}^2). \tag{A5}$$

This implies that

$$\begin{aligned}\langle \boldsymbol{k}'|L\phi|\boldsymbol{k}\rangle &\approx -\frac{5}{16m^2}\,\mathrm{i}\langle \boldsymbol{k}'|\boldsymbol{p}^2\phi - \phi\boldsymbol{p}^2|\boldsymbol{k}\rangle \\ &= -\frac{5}{16m^2}\langle \boldsymbol{k}'|\boldsymbol{p}\cdot\nabla\phi + \nabla\phi\cdot\boldsymbol{p}|\boldsymbol{k}\rangle.\end{aligned} \tag{A6}$$

This is an alternative proof of (4.1) in the text.

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